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Compactness in spaces of inner regular measures and a general Portmanteau lemma[☆]

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ABSTRACT

This paper may be understood as a continuation of Topsoe's seminal paper [F. Topsoe, Compactness in spaces of measures, *Studia Math.* 36 (1970) 195–212] to characterize, within an abstract setting, compact subsets of finite inner regular measures w.r.t. the weak topology. The new aspect is that neither assumptions on compactness of the inner approximating lattices nor nonsequential continuity properties for the measures will be imposed. As a providing step also a generalization of the classical Portmanteau lemma will be established. The obtained characterizations of compact subsets w.r.t. the weak topology encompass several known ones from literature. The investigations rely basically on the inner extension theory for measures which has been systemized recently by König [H. König, *Measure and Integration*, Springer, Berlin, 1997; H. König, On the inner Daniell-Stone and Riesz representation theorems, *Doc. Mat.* 5 (2000) 301–315; H. König, *Measure and integration: An attempt at unified systematization*, *Rend. Istit. Mat. Univ. Trieste* 34 (2002) 155–214].

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0. Introduction

The most influential result concerning compactness of spaces of measures has been presented by Prokhorov [17] for probability Borel-measures on Polish spaces. His equivalent characterization of relative compactness by uniform tightness has turned out to be an important tool to check convergence in law for many stochastic processes. Therefore nowadays this criterion may be found in most of standard textbooks of probability theory (see e.g. [3,4,7,16]). Another characterization of compact sets of Borel probability measures on Polish spaces has been shown by Huber and Strassen [8] in terms of a continuity property that the upper envelopes of these sets satisfy. Prokhorov as well as Huber and Strassen used the so called topology of weak convergence which is derived from the weak $*$ topology on the algebraic dual of the space of bounded continuous mappings. Of course this topology may be extended to spaces of finite Baire-measures on general topological spaces. This has been done by Varadarajan [22] who has also found an equivalent characterization for compact sets. However the topology of weak convergence relies on hidden regularity properties. Finite Baire-measures are inner regular w.r.t. the functionally closed sets, and in the special context of metrizable spaces they coincide with the finite Borel-measures, being inner regular w.r.t. the closed subsets. But in general finite Borel-measures are not inner regular w.r.t. the closed subsets. So for spaces of such measures the topology of weak convergence is not a reasonable concept because the measures are not uniquely determined by the restrictions of the integrals to bounded continuous mappings. Furthermore it seems to be necessary to impose regularity for the measures to find tractable extensions of the topology of weak convergence.

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In recent years, many issues of mathematical finance like portfolio selection, efficient hedging, dynamical investment and consumption have been investigated within a setting of the following type of optimization problems

$$\text{minimize } h(k) := \sup_{\mu \in \Delta} \left(\int g(k, \cdot) d\mu - \gamma(\mu) \right),$$

where $g : K \times \Delta \rightarrow \mathbb{R}$ and $\gamma : \Delta \rightarrow]-\infty, \infty]$ with K being a convex subset of a real vector space, and Δ denotes a set of inner regular finite measures. Typically, minimax arguments will be needed to employ the duality techniques from convex analysis (cf. e.g. [6,18,23]). Since in general the measures occurring in these problems are neither Baire- nor Borel-measures, it seems to be convenient to find characterizations of compactness of inner regular measures within an abstract setting.

Fresh ideas had been presented by Topsoe in two seminal publications (cf. [19,20]). The framework is based on a pair \mathcal{S}, \mathcal{G} of lattices on an abstract set Ω , where \mathcal{S} is stable under countable intersections. One may think of \mathcal{G} as a topology and \mathcal{S} as the set of closed or closed compact sets. It is known from extension results (e.g. [10, Theorem 6.31]) that a finite measure on the σ -algebra $\sigma(\mathcal{S})$ generated by \mathcal{S} which is inner regular w.r.t. \mathcal{S} may be extended to the σ -algebra $\sigma(\mathcal{STS})$ generated by the transporter $\mathcal{STS} := \{A \subseteq \Omega \mid A \cap B \in \mathcal{S} \text{ for every } B \in \mathcal{S}\}$. In particular for \mathcal{S} containing the closed compact subsets, we obtain extensions to Radon measures. So, assuming that the complements of the members of \mathcal{G} are contained in \mathcal{STS} , Topsoe considered the space of finite measures on $\sigma(\mathcal{STS})$ which are inner regular w.r.t. \mathcal{S} , and he equipped this space with the coarsest topology such that $Q \mapsto Q(G)$ is lower semicontinuous for $G \in \mathcal{G} \cup \{\Omega\}$, and even continuous in the case of $G = \Omega$. It is a generalization of the topology of weak convergence in view of the classical Portmanteau lemma, and so he called it weak topology. Under the assumption that disjoint sets from \mathcal{S} may be separated by disjoint sets from \mathcal{G} , he succeeded in giving a general characterization of relatively compact subsets in two cases. Firstly, if \mathcal{S} is semicompact, and secondly for subsets in the topological subspace of finite measures \mathcal{P} satisfying $\inf_{A \in M} P(A) = P(B)$ for every downward directed family M in \mathcal{S} with $\bigcap_{A \in M} A = B \in \mathcal{S}$ if $\Omega \in \mathcal{S}$ (cf. [19]).

This paper takes up the investigations by Topsoe. The aim is to characterize the relatively compact sets of finite measures which are inner regular w.r.t. the lattice \mathcal{S} without further assumptions. So \mathcal{S} need not to be semicompact, and the nonsequential continuity property of the measures will be not assumed.

The paper is organized as follows. In the next section some basic concepts and results from abstract measure and integration theory will be recalled. Besides a useful general Daniell-Stone representation theorem some inner extension results will be reviewed. Afterwards we shall introduce in Section 2 the weak topology on spaces $\mathcal{M}_f(\Omega, \mathcal{S})$ of finite measures which are inner regular w.r.t. lattices \mathcal{S} containing the sample space Ω and being stable under countable intersections. However, we shall propose a slightly different approach which does not rely on an additional lattice \mathcal{G} in general. It coincides with Topsoe's suggestion in many relevant cases, in particular under his separation property. In particular his results are exactly formulated w.r.t. the weak topology. Moreover, we shall achieve to relax Topsoe's separation property. This offers the opportunity to describe the topology in terms of Topsoe's approach whenever it is convenient. For instance in many cases we might choose the partner lattices \mathcal{G} as being generated by weak upper level sets of special kind of function systems, with the lattices of functionally open subsets of Hausdorff spaces as prominent examples. Such function systems suggest to define topologies on $\mathcal{M}_f(\Omega, \mathcal{S})$ analogously to the classical topology of weak convergence. The subject of Section 3 is to compare it with the weak topology. The investigations lead to a generalized version of the Portmanteau lemma, in particular $\mathcal{M}_f(\Omega, \mathcal{S})$ is a completely regular Hausdorff space without imposing the adaption of Topsoe's separation property. The results from Sections 2, 3 will be used to show several characterizations of compact subsets w.r.t. the weak topology on $\mathcal{M}_f(\Omega, \mathcal{S})$. Suitable specializations retain the contributions by Varadarajan as well as Huber and Strassen.

1. Notations and preliminaries

Let us begin with recalling some basic notions from abstract measure and integration theory. The reader is referred to the monograph by König (cf. [10], overview in [14]) for a comprehensive account.

Let Ω be a nonvoid set. Nonvoid collections of subsets of Ω are called *lattices* if they are stable under finite unions and intersections. For a lattice \mathcal{S} on Ω , the symbol $\sigma(\mathcal{S})$ stands for the σ -algebra on Ω generated by \mathcal{S} .

A set function $\phi : \mathcal{S} \rightarrow [0, \infty]$ on a lattice \mathcal{S} is said to be *isotone* if $\phi(A) \leq \phi(B)$ holds for every pair $A, B \in \mathcal{S}$ with $A \subseteq B$, and it is defined to be *modular* if $\phi(A \cup B) + \phi(A \cap B) = \phi(A) + \phi(B)$ for $A, B \in \mathcal{S}$. We shall call an isotone set function ϕ on the lattice \mathcal{S} to be *downward continuous at A* if $A \in \mathcal{S}$, and $\inf_n \phi(A_n) = \phi(A)$ whenever $(A_n)_n$ is an antitone sequence in \mathcal{S} with $\bigcap_{n=1}^\infty A_n = A$. If it is downward continuous at each $A \in \mathcal{S}$, we shall say that it is *downward continuous*.

Another important concept within measure theory is regularity. Setting $\inf \emptyset := \infty$, $\sup \emptyset := 0$ an isotone set function ϕ on a lattice \mathcal{S} is said to be *inner/outer regular w.r.t. T* if $T \subseteq \mathcal{S}$, and

$$\phi(A) = \sup_{A \supseteq T \in \mathcal{T}} \phi(T) \quad \text{resp.} \quad \phi(A) = \inf_{A \subseteq T \in \mathcal{T}} \phi(T)$$

for all $A \in \mathcal{S}$. An isotone set function ϕ on a lattice \mathcal{S} which is stable under countable intersections is defined to be an *inner precontent/inner premeasure* if $\emptyset \in \mathcal{S}$ with $\phi(\emptyset) = 0$, and if it can be extended to a content/measure on an algebra/ σ -algebra which is inner regular w.r.t. \mathcal{S} . We shall need the following extension results by König.

Proposition 1.1. Let ϕ be a bounded isotone, modular set function ϕ on a lattice S which is stable under countable intersections, and which contains \emptyset with $\phi(\emptyset) = 0$. Then we have:

- (1) ϕ is an inner premeasure if it satisfies the following conditions:
 - (a) ϕ is downward continuous at \emptyset ;
 - (b) $\phi(B) \leq \sup_{B \setminus A \supseteq D \in S} \phi(D) + \phi(A)$ for $A, B \in S$ with $A \subseteq B$.
- (2) Let $\mathcal{T} \subseteq \{\Omega \setminus B \mid A \cap B \in S \text{ for every } A \in S\}$ be a lattice with $\emptyset \in \mathcal{T}$ such that two disjoint sets from S may be separated by two disjoint sets from \mathcal{T} . Then the mapping

$$\widehat{\phi} : S \rightarrow \mathbb{R}, \quad A \mapsto \inf_{A \subseteq G \in \mathcal{T}} \sup_{G \supseteq B \in S} \phi(B)$$

is an inner precontent. It is even an inner premeasure if $\widehat{\phi}$ is downward continuous at \emptyset .

Statement (1) follows from Theorem 6.31, 5'), and discussion afterwards in [10] (see also Theorem 3.5 in [14]). Drawing on Proposition 3.3 in [13], statement (2) may be concluded from (1).

The following type of measure extension will turn out to be useful too.

Proposition 1.2. Let $S \supseteq \widetilde{S}$ be lattices on an abstract set Ω which are stable under countable intersections containing \emptyset , $\Omega \in \widetilde{S}$, and let P be a finite measure on $\sigma(\widetilde{S})$ which is inner regular w.r.t. \widetilde{S} . Then there exists a unique finite measure Q on $\sigma(S)$ which is inner regular w.r.t. S and extends P if the following properties are fulfilled:

- (1) for every antitone sequence $(A_n)_n$ in S with $\bigcap_{n=1}^{\infty} A_n = \emptyset$ there exists an antitone sequence $(B_n)_n$ in $\sigma(\widetilde{S})$ with $A_n \subseteq B_n$ for each n and $\bigcap_{n=1}^{\infty} B_n = \emptyset$,
- (2) disjoint sets A_1, A_2 from S may be separated by disjoint $B_1, B_2 \in \sigma(\widetilde{S})$.

A proof of Proposition 1.2 may found in [1, Theorem 3.3] or in [10, Theorem 19.11].

We shall make also use of a general inner Daniell-Stone representation result. Let us recall that a function system $E \subseteq [0, \infty]^\Omega$ is called a *Stonean lattice cone* if for $X, Y \in E$ and $\lambda \geq 0$, $t > 0$ the mappings $X + Y$, $\min\{X, Y\}$, $\max\{X, Y\}$ as well as λX , $\min\{X, t\}$, $\max\{X - t, 0\}$ belong to E . A functional $I : E \rightarrow \mathbb{R}$ is defined to be *isotone and positive-linear* if $I(X) \leq I(Y)$ for $X \leq Y$ and $I(\lambda X) = \lambda I(X)$ as well as $I(X + Y) = I(X) + I(Y)$ for $X, Y \in E$, $\lambda \geq 0$. The announced inner Daniell-Stone theorem may be found in [12] (Theorem 2.5 with Theorem 1.3 and Theorem 4.2 in [11]).

Proposition 1.3. Let $I : E \rightarrow \mathbb{R}$ be an isotone and positive-linear functional on a Stonean lattice cone $E \subseteq [0, \infty]^\Omega$ which contains the nonnegative constants. Furthermore let S be defined to consist of all $\bigcap_{n=1}^{\infty} X_n^{-1}([x_n, \infty])$ with $X_n \in E$ and $x_n > 0$. Moreover, it is supposed that for any $A \in S$ there is some sequence $(A_n)_n$ in S such that $\Omega \setminus A = \bigcup_{n=1}^{\infty} A_n$ holds.

Then there exists a finite measure P on $\sigma(S)$ which is inner regular w.r.t. S and satisfies $\int X dP = I(X)$ for every $X \in E$ if and only if $\inf_n I(X_n) = 0$ for $X_n \searrow 0$ and $\sup_n I(Y_n) = I(Y)$ for $Y_n \nearrow Y$. In this case all representing finite measures are inner regular w.r.t. S and coincide.

2. Weak topologies on spaces of inner regular finite measures

Let S be a lattice on a nonvoid set Ω satisfying:

- (2.1) $\emptyset, \Omega \in S$;
- (2.2) S stable under countable intersections.

We shall consider the set $\mathcal{M}_f(\Omega, S)$ gathering all finite measures on $\sigma(S)$ which are inner regular w.r.t. S . It will be equipped with the coarsest topology τ_w such that the mapping $\psi_A : \mathcal{M}_f(\Omega, S) \rightarrow \mathbb{R}$, $Q \mapsto Q(A)$, is upper semicontinuous for each $A \in S$, and such that ψ_Ω is continuous. We may describe τ_w also by the basic neighbourhood system consisting of

$$N_w(P, A_1, \dots, A_n, \varepsilon) := \{Q \in \mathcal{M}_f(\Omega, S) \mid |P(\Omega) - Q(\Omega)| < \varepsilon, Q(A_i) < P(A_i) + \varepsilon, i = 1, \dots, n\}$$

for $P \in \mathcal{M}_f(\Omega, S)$, $n \in \mathbb{N}$, $A_1, \dots, A_n \in S$, $\varepsilon > 0$. Historically, for finite Baire-measures Alexandroff (cf. [2]) introduced the topology induced by the weak convergence. To recall weak convergence means that a net $(Q)_{j \in J}$ of finite Baire-measures converges to a finite Baire-measure Q if $(\int X dQ_j)_{j \in J}$ converges to $\int X dQ$ for every bounded continuous X . This topology coincides with the usual topology used for finite Borel-measures in the context of metrizable topologies. Recall that the functionally closed subsets are exactly the subsets of the form $X^{-1}(\{0\})$, where X denotes a real-valued continuous mapping. The functionally open subsets are the complements of functionally closed ones. Since finite Baire-measures are inner regular w.r.t. the functionally closed subsets (cf. [10, Addendum 8.5]), and since finite Borel-measures on metric spaces are inner regular w.r.t. the closed subsets, we can recognize by classical Portmanteau lemma (e.g. [2, p. 180]) that in the topological

context this classical approach coincides with τ_w . Notice that the set of functionally closed subset is stable under countable intersections since $\sum_{n=1}^{\infty} \frac{X_n}{2^n}$ is a bounded continuous mapping for any uniformly bounded sequence $(X_n)_n$ of continuous functions.

Throughout this paper we shall call τ_w the *weak topology*. This is in accordance with Topsoe's suggestion to define weak topologies for finite inner regular measures within an abstract framework, as will become clear after a comparison with his approach developed in [19]. It may be characterized as follows. Let $\tilde{\mathcal{S}} \subseteq \mathcal{S}$ be some additional lattice satisfying:

$$(2.3) \quad \emptyset, \Omega \in \tilde{\mathcal{S}};$$

$$(2.4) \quad \tilde{\mathcal{S}} \text{ is stable under countable intersections.}$$

Setting $\mathcal{G}_{\tilde{\mathcal{S}}} := \{\Omega \setminus A \mid A \in \tilde{\mathcal{S}}\}$, Topsoe defines the topology $\tau_{\mathcal{G}_{\tilde{\mathcal{S}}}}$ to be the coarsest topology on $\mathcal{M}_f(\Omega, \mathcal{S})$ such that for each $G \in \mathcal{G}_{\tilde{\mathcal{S}}}$ the mapping $\psi_G : \mathcal{M}_f(\Omega, \mathcal{S}) \rightarrow \mathbb{R}, Q \mapsto Q(G)$, is lower semicontinuous, and such that ψ_{Ω} is continuous. Equivalently, the mappings $\psi_A : \mathcal{M}_f(\Omega, \mathcal{S}) \rightarrow \mathbb{R}, Q \mapsto Q(A)$, should be upper semicontinuous for each $A \in \tilde{\mathcal{S}}$, and even continuous in the case $A = \Omega$. In general $\tau_{\mathcal{G}_{\tilde{\mathcal{S}}}} \subseteq \tau_w$. In the following we shall introduce separation properties for the involved lattices such that these topologies are identical. First of all the topology $\tau_{\mathcal{G}_{\tilde{\mathcal{S}}}} \subseteq \tau_w$ should fulfill the topological T_0 -separation, i.e. for different P, Q from $\mathcal{M}_f(\Omega, \mathcal{S})$ the restrictions to $\mathcal{G}_{\tilde{\mathcal{S}}}$ should differ too. This may be achieved by the condition

$$(2.5) \quad \text{disjoint sets } A_1, A_2 \text{ from } \mathcal{S} \text{ may be separated by disjoint } B_1, B_2 \in \sigma(\tilde{\mathcal{S}}).$$

Condition (2.5) is redundant for $\tilde{\mathcal{S}} = \mathcal{S}$.

Lemma 2.1. *Let $P, Q \in \mathcal{M}_f(\Omega, \mathcal{S})$ coincide on $\tilde{\mathcal{S}}$. Then $P = Q$ holds under (2.1)–(2.5), and moreover $P(A) = Q(A) = \inf_{A \subseteq B \in \sigma(\tilde{\mathcal{S}})} P(B)$ for any $A \in \mathcal{S}$.*

Proof. By assumption $P|_{\tilde{\mathcal{S}}} = Q|_{\tilde{\mathcal{S}}}$, and therefore P, Q coincide on the algebra generated by $\tilde{\mathcal{S}}$ since this algebra just gathers the disjoint unions of differences of sets from $\tilde{\mathcal{S}}$. In view of the classical extension result for nonnegative σ -additive set functions on algebras $P|\sigma(\tilde{\mathcal{S}}) = Q|\sigma(\tilde{\mathcal{S}})$. Now let us define $\phi : \mathcal{S} \rightarrow [0, \infty]$ by $\phi(A) := \inf_{A \subseteq G \in \sigma(\tilde{\mathcal{S}})} P(G)$. Obviously, $\phi \geq P|_{\mathcal{S}}$. Since P, Q coincide on $\sigma(\tilde{\mathcal{S}})$, and since they are inner regular w.r.t. \mathcal{S} , it suffices to prove $\phi(A) \leq P(A)$ for any $A \in \mathcal{S}$. For this purpose let $A \in \mathcal{S}$, and let $D \subseteq \Omega \setminus A$ belong to \mathcal{S} . By condition (2.5) we may find disjoint $G_1, G_2 \in \sigma(\tilde{\mathcal{S}})$ with $D \subseteq G_1$ and $A \subseteq G_2$. Furthermore for any positive ε we may find by definition of ϕ some $G \in \sigma(\tilde{\mathcal{S}})$ with $D \cup A \subseteq G$ and $\phi(D \cup A) + \varepsilon \geq P(G)$. Therefore

$$\phi(D \cup A) + \varepsilon \geq P(G \cap G_1) + P(G \cap G_2) \geq \phi(D) + \phi(A).$$

In particular

$$P(D) \leq \phi(D) \leq \phi(D \cup A) - \phi(A) \leq P(\Omega) - \phi(A),$$

and hence, since P is inner regular w.r.t. \mathcal{S} ,

$$P(\Omega) - P(A) = \sup_{\Omega \setminus A \supseteq D \in \mathcal{S}} P(D) \leq P(\Omega) - \phi(A),$$

which completes the proof. \square

The next condition ensures that restrictions of measures from $\mathcal{M}_f(\Omega, \mathcal{S})$ to $\sigma(\tilde{\mathcal{S}})$ are inner regular w.r.t. $\tilde{\mathcal{S}}$:

$$(2.6) \quad \text{for disjoint } A_1 \in \mathcal{S} \text{ and } A_2 \in \tilde{\mathcal{S}} \text{ there exists some } B \in \tilde{\mathcal{S}} \text{ with } A_1 \subseteq B \subseteq \Omega \setminus A_2.$$

Condition (2.6) is trivial if $\tilde{\mathcal{S}} = \mathcal{S}$.

Lemma 2.2. *Let conditions (2.1)–(2.4), (2.6) be fulfilled. Then for $P \in \mathcal{M}_f(\Omega, \mathcal{S})$ the restriction $P|\sigma(\tilde{\mathcal{S}})$ is inner regular w.r.t. $\tilde{\mathcal{S}}$, and therefore outer regular w.r.t. $\mathcal{G}_{\tilde{\mathcal{S}}}$.*

Proof. Let $P \in \mathcal{M}_f(\Omega, \mathcal{S})$, and let $\phi := P|_{\tilde{\mathcal{S}}}$, which defines an isotone modular set function satisfying $\phi(\emptyset) = 0$ and $\inf_n \phi(A_n) = 0$ for any antitone sequence $(A_n)_n$ in $\tilde{\mathcal{S}}$ with $\bigcap_{n=1}^{\infty} A_n = \emptyset$. Next, let $A, B \in \tilde{\mathcal{S}}$ with $A \subseteq B$. Since P is inner regular w.r.t. \mathcal{S} we may find for arbitrary $\varepsilon > 0$ some $D \in \mathcal{S}$ with $D \subseteq B \setminus A$ such that $P(B \setminus A) - \varepsilon < P(D)$. By assumption $A, B \in \tilde{\mathcal{S}}$ so that condition (2.6) gives some $\tilde{A} \in \tilde{\mathcal{S}}$ with $D \subseteq \tilde{A} \subseteq B \setminus A$. Thus

$$\phi(B) - \varepsilon < \phi(\tilde{A}) + \phi(A) \leq \sup_{B \setminus A \supseteq \tilde{A} \in \tilde{\mathcal{S}}} \phi(\tilde{A}) + \phi(A).$$

Therefore ϕ is an inner premeasure due to Proposition 1.1(1), and there exists a measure Q on $\sigma(\tilde{S})$ which is inner regular w.r.t. \tilde{S} and extends ϕ . As a further consequence we have $Q \leq P|_{\sigma(\tilde{S})}$ and $Q(\Omega) = P(\Omega)$. This implies $Q = P|_{\sigma(\tilde{S})}$, and completes the proof because the outer regularity w.r.t. $\mathcal{G}_{\tilde{S}}$ follows from the inner regularity w.r.t. \tilde{S} by using complements. \square

Combining Lemmata 2.1, 2.2, it turns out that $\tau_{\mathcal{G}_{\tilde{S}}}$ and τ_w coincide under additional conditions (2.5), (2.6).

Theorem 2.3. *If the conditions (2.1)–(2.6) are fulfilled, then $\tau_{\mathcal{G}_{\tilde{S}}} = \tau_w$.*

Proof. It remains to show $\tau_{\mathcal{G}_{\tilde{S}}} \supseteq \tau_w$. For this purpose let $(P_j)_j$ denote a net in $\mathcal{M}_f(\Omega, S)$ which converges to $P \in \mathcal{M}_f(\Omega, S)$ w.r.t. $\tau_{\mathcal{G}_{\tilde{S}}}$, and let $A \in S$. In view of Lemmata 2.1, 2.2, we may find for arbitrary $\varepsilon > 0$ some $G \in \mathcal{G}_{\tilde{S}}$ with $A \subseteq G$ and $P(A) + \varepsilon > P(G)$. Then there exists by condition (2.6) some $B \in \tilde{S}$ with $A \subseteq B \subseteq G$. Therefore by definition of $\tau_{\mathcal{G}_{\tilde{S}}}$

$$P(A) + \varepsilon > P(B) \geq \limsup_j P_j(B) \geq \limsup_j P_j(A),$$

which implies that $(P_j)_j$ converges to P w.r.t. τ_w . The proof is now complete. \square

In order to show that $(\mathcal{M}_f(\Omega, S), \tau_w)$ is a regular Hausdorff space we impose the following condition.

(2.7) Disjoint sets A_1, A_2 from \tilde{S} may be separated by disjoint $G_1, G_2 \in \mathcal{G}_{\tilde{S}}$.

Proposition 2.4. *Let the assumptions (2.1)–(2.7) be satisfied. Then $(\mathcal{M}_f(\Omega, S), \tau_{\mathcal{G}_{\tilde{S}}}) = (\mathcal{M}_f(\Omega, S), \tau_w)$ is a regular Hausdorff space.*

Proof. In view of Theorem 2.3 $\tau_{\mathcal{G}_{\tilde{S}}} = \tau_w$.

For different $P, Q \in \mathcal{M}_f(\Omega, S)$ there exists some $A \in S$ with $P(A) \neq Q(A)$ due to inner regularity of P and Q . Therefore $P \notin N_w(Q, A, \varepsilon)$ for sufficient small $\varepsilon > 0$ if $P(A) > Q(A)$, and $Q \notin N_w(P, A, \varepsilon)$ for sufficient small $\varepsilon > 0$ in the case of $Q(A) > P(A)$. Thus $(\mathcal{M}_f(\Omega, S), \tau_w)$ is a T_1 -space (see [5, p. 36]), meaning that the regularity of τ_w is left to show (cf. [5, p. 38]).

For this purpose, notice that the topology $\tau_{\mathcal{G}_{\tilde{S}}}$ may be described by the basic neighbourhood system consisting of

$$N(P, A_1, \dots, A_n, \varepsilon) := \{Q \in \mathcal{M}_f(\Omega, S) \mid |P(\Omega) - Q(\Omega)| < \varepsilon, Q(A_i) < P(A_i) + \varepsilon, i = 1, \dots, n\}$$

for $P \in \mathcal{M}_f(\Omega, S)$, $n \in \mathbb{N}$, $A_1, \dots, A_n \in \tilde{S}$, $\varepsilon > 0$. Let us fix some $P \in \mathcal{M}_f(\Omega, S)$ and a neighbourhood $N(P, A_1, \dots, A_n, \varepsilon)$. Due to condition (2.6) and Lemma 2.2 there exists for each A_i a set $G_i \in \mathcal{G}_{\tilde{S}}$ which encloses A_i and satisfies $|P(G_i) - P(A_i)| < \frac{\varepsilon}{2}$. By (2.7) we may find for every A_i sets $H_i \in \mathcal{G}_{\tilde{S}}$ and $B_i \in \tilde{S}$ with $A_i \subseteq H_i \subseteq B_i \subseteq G_i$. Now let $(Q_j)_{j \in J}$ denote a net in $N(P, B_1, \dots, B_n, \frac{\varepsilon}{2})$ which converges to some Q w.r.t. $\tau_{\mathcal{G}_{\tilde{S}}}$. Then we may conclude for each $i \in \{1, \dots, n\}$

$$Q(A_i) \leq Q(H_i) \leq \liminf_j Q_j(H_i) \leq \liminf_j Q_j(B_i) \leq P(B_i) + \frac{\varepsilon}{2} < P(A_i) + \varepsilon.$$

Hence the closure of $N(P, B_1, \dots, B_n, \frac{\varepsilon}{2})$ is contained in $N(P, A_1, \dots, A_n, \varepsilon)$, which completes the proof. \square

Setting $\tilde{S} := S$, conditions (2.5), (2.6) are redundant, so that $\tau_{\mathcal{G}_{\tilde{S}}} = \tau_w$ due to Theorem 2.3. Therefore we obtain immediately the following corollary of Proposition 2.4.

Corollary 2.5. *$(\mathcal{M}_f(\Omega, S), \tau_w)$ is a regular Hausdorff space if disjoint sets from S may be separated by disjoint sets from $\{\Omega \setminus A \mid A \in S\}$.*

Remark 2.6. The conditions (2.5)–(2.7) are implied by

(Top) disjoint sets A_1, A_2 from S may be separated by disjoint $G_1, G_2 \in \mathcal{G}_{\tilde{S}}$.

This is just the initial separation property that Topsoe used in [19]. So under condition (Top) τ_w is just the topology introduced by Topsoe.

Let us give some examples.

Examples 2.7. Consider the following specializations of \tilde{S}, S :

- (1) *Strong topology*
 $\tilde{\mathcal{S}} = \mathcal{S} = \sigma(\mathcal{S})$.
 Condition (Top) is always satisfied.
- (2) *Topology of weak convergence for finite Baire-measures*
 $\tilde{\mathcal{S}} = \mathcal{S}$ set of functionally closed subsets of a Hausdorff space.
 Condition (Top) is fulfilled.
- (3) *Weak topologies for finite Borel-measures*
 \mathcal{S} set of closed subsets of a normal Hausdorff space, and either $\tilde{\mathcal{S}} = \mathcal{S}$ or $\tilde{\mathcal{S}}$ is defined to consist of all functionally closed subsets.
 Condition (Top) is guaranteed due to definition of normal topologies and Urysohn's lemma.

For the investigations later on the following continuity condition will play an important role.

- (2.8) For every antitone sequence $(A_n)_n$ in \mathcal{S} with $\bigcap_{n=1}^{\infty} A_n = \emptyset$ there exists an antitone sequence $(B_n)_n$ in $\sigma(\tilde{\mathcal{S}})$ with $A_n \subseteq B_n$ for each n and $\bigcap_{n=1}^{\infty} B_n = \emptyset$.

Next we want to avoid separation condition (2.7). Instead we consider the case that $\tilde{\mathcal{S}}$ consists of countable intersections of level sets $X^{-1}([x, \infty[)$ of mappings X from a Stonean lattice cone $E \subseteq [0, \infty[^\Omega$. In order to obtain properties for τ_w in this situation we shall provide us in the following section with a general Portmanteau lemma.

3. A general Portmanteau lemma

Throughout this section we shall assume that there is some Stonean lattice cone $E \subseteq [0, \infty[^\Omega$ with $1 \in E$ such that $\tilde{\mathcal{S}} := \{\bigcap_{n=1}^{\infty} X_n^{-1}([x_n, \infty[) \mid X_n \in E, x_n > 0\} \subseteq \mathcal{S}$. Additionally all members of E should be bounded, and the further condition

- (3.1) $\sup X - X \in E$ for every $X \in E$

should be fulfilled. Notice that under (3.1) we may find for any $A \in \tilde{\mathcal{S}}$ a sequence $(A_n)_n$ in $\tilde{\mathcal{S}}$ such that $\Omega \setminus A = \bigcup_{n=1}^{\infty} A_n$ holds.

The function system E induces a topology $\tau_{w,E}$ on $\mathcal{M}_f(\Omega, \mathcal{S})$ defined by the basic neighbourhood system consisting of

$$N_E(P, X_1, \dots, X_n, \varepsilon) := \left\{ Q \in \mathcal{M}_f(\Omega, \mathcal{S}) \mid \left| \int X_i dQ - \int X_i dP \right| < \varepsilon, i = 1, \dots, n \right\}$$

for $P \in \mathcal{M}_f(\Omega, \mathcal{S})$, $n \in \mathbb{N}$, $\varepsilon > 0$ and $X_1, \dots, X_n \in E$. In view of the inner Daniell-Stone Theorem 1.3 measures on $\sigma(\tilde{\mathcal{S}})$ whose integrals coincide on E are identical. Therefore, due to Lemma 2.1, measures from $\mathcal{M}_f(\Omega, \mathcal{S})$ are uniquely determined by the restrictions of their integrals to E provided that (2.5) is satisfied. In this case $\tau_{w,E}$ is obviously Hausdorff. Moreover, transferring the proof of Theorem II.1 in [22] verbatim, we obtain that under (2.5) $(\mathcal{M}_f(\Omega, \mathcal{S}), \tau_{w,E})$ is even a completely regular Hausdorff space. In the next step we want to compare the topologies τ_w and $\tau_{w,E}$. The investigations lead to a general Portmanteau lemma.

Theorem 3.1. Let $Q \in \mathcal{M}_f(\Omega, \mathcal{S})$, let $(Q_j)_{j \in J}$ be a net in $\mathcal{M}_f(\Omega, \mathcal{S})$, and consider the following statements:

- (1) $\lim_j Q_j(\Omega) = Q(\Omega)$ and $\limsup_j Q_j(A) \leq Q(A)$ for all $A \in \mathcal{S}$.
- (2) $\lim_j Q_j(\Omega) = Q(\Omega)$ and $\liminf_j Q_j(G) \geq Q(G)$ for all $G \in \{\Omega \setminus A \mid A \in \mathcal{S}\}$.
- (3) $\lim_j Q_j(\Omega) = Q(\Omega)$ and $\limsup_j Q_j(A) \leq Q(A)$ for all $A \in \tilde{\mathcal{S}}$.
- (4) $\lim_j Q_j(\Omega) = Q(\Omega)$ and $\liminf_j Q_j(G) \geq Q(G)$ for all $G \in \{\Omega \setminus A \mid A \in \tilde{\mathcal{S}}\}$.
- (5) $\lim_j \int X dQ_j = \int X dQ$ for all $X \in E$.

Then under assumptions (2.1), (2.2), (2.5), (3.1) (1) \Rightarrow (3) \Leftrightarrow (5) and the equivalences (1) \Leftrightarrow (2) as well as (3) \Leftrightarrow (4) hold. If in addition assumption (2.6) is valid, then all statements are equivalent.

Proof. Let (2.1), (2.2), (2.5), (3.1) be satisfied. Then the equivalences (1) \Leftrightarrow (2) and (3) \Leftrightarrow (4) are obvious, also implication (1) \Rightarrow (3) by $\tilde{\mathcal{S}} \subseteq \mathcal{S}$.

Proof of (5) \Rightarrow (3):

Let $A \in \tilde{\mathcal{S}}$ with indicator mapping 1_A . Since $1 \in E$, statement (5) implies $\lim_j Q_j(\Omega) = Q(\Omega)$. Moreover, it is known that there exists some antitone sequence $(X_n)_n$ in E with $1_A = \inf_n X_n$ (cf. [11, Proposition 3.2]). Therefore $\psi_A = \inf_n \psi_{X_n}$, where

$$\psi_A : \mathcal{M}_f(\Omega, \mathcal{S}) \rightarrow \mathbb{R}, \quad Q \mapsto Q(A), \quad \text{and} \quad \psi_{X_n} : \mathcal{M}_f(\Omega, \mathcal{S}) \rightarrow \mathbb{R}, \quad Q \mapsto \int X_n dQ \quad (n \in \mathbb{N}).$$

In particular ψ_A is upper semicontinuous w.r.t. $\tau_{w,E}$, and thus $\limsup_j Q_j(A) \leq Q(A)$ follows immediately from statement (5).

Proof of (3) \Rightarrow (5):

Let $M_b(\Omega)$ denote the space of all bounded real-valued mappings on Ω . It will be equipped with the supremum norm $\|\cdot\|_\infty$.

Let $X \in E$. Since $X \in M_b(\Omega)$ with $X^{-1}([x, \infty]) \in \tilde{\mathcal{S}}$ for $x > 0$ we may approximate it uniformly by an isotone sequence $(X_n)_n$ of nonnegative functions with finite range and level sets $X_n^{-1}([x, \infty])$ ($x > 0, n \in \mathbb{N}$) belonging to $\tilde{\mathcal{S}}$ (cf. [10, Proposition 22.1]). We may find for each $n \in \mathbb{N}$ some $\lambda_1, \dots, \lambda_{r_n} > 0$ and $A_1, \dots, A_{r_n} \in \tilde{\mathcal{S}}$ such that for any $P \in \{Q, Q_j \mid j \in J\}$ the integral $\int X_n dP$ may be described by $\sum_{i=1}^{r_n} \lambda_i P(A_i)$ (cf. [10, Properties 11.8]). Thus statement (3) implies

$$\limsup_j \int X dQ_j \leq \limsup_j \left(Q_j(\Omega) \|X - X_n\|_\infty + \int X_n dQ_j \right) \leq Q(\Omega) \|X - X_n\|_\infty + \int X_n dQ$$

for every $n \in \mathbb{N}$. Hence $\limsup_j \int X dQ_j \leq \int X dQ$ by monotone convergence. Since $\sup X - X \in E$ due to (3.1), we may employ the same line of reasoning to obtain $\limsup_j \int (\sup X - X) dQ_j \leq \int (\sup X - X) dQ$. This shows $\lim_j \int X dQ_j = \int X dQ$.

Finally, additional assumption (2.6) forces implication (3) \Rightarrow (1) due to Theorem 2.3. This means that all statements are equivalent, and the proof is complete. \square

As a consequence of the Portmanteau lemma and the discussion on $\tau_{w,E}$ we can emphasize the following property of τ_w .

Proposition 3.2. $(\mathcal{M}_f(\Omega, \mathcal{S}), \tau_w)$ is a completely regular Hausdorff space if the conditions (2.1), (2.2), (2.5), (2.6) and (3.1) are valid.

Remark 3.3. In order to find for the lattice \mathcal{S} a lattice $\tilde{\mathcal{S}}$ and the Stonean lattice cone E in the Portmanteau Lemma 3.1, a first attempt might be to choose the function system E defined to consist of the bounded nonnegative $X \in \mathbb{R}^\Omega$ with $X^{-1}([x, \infty])$, $X^{-1}((-\infty, x]) \in \mathcal{S}$ for $x > 0$. It is indeed a Stonean lattice cone which satisfies (3.1). Then one has to look whether in addition the assumptions (2.5) and (2.6) are satisfied with $\tilde{\mathcal{S}} := \{\bigcap_{n=1}^\infty X_n^{-1}([x_n, \infty]) \mid X_n \in E, x_n > 0\}$. For prominent applications of this line of reasoning let (Ω, τ_Ω) be a topological space:

- (1) If \mathcal{S} is the set of the functionally closed subsets, then E is the set of nonnegative bounded continuous mappings on Ω with $\tilde{\mathcal{S}} = \mathcal{S}$. Then the assumptions (2.5), (2.6) are satisfied. Therefore Theorem 3.1 retains the classical Portmanteau lemma for finite Baire-measures, and for finite Borel-measures in the case of metrizable τ_Ω . Moreover, the classical Portmanteau lemma may be extended to finite Borel-measures if τ_Ω is perfectly normal (notice [5, 1.5.19]).
- (2) If \mathcal{S} is the set of the closed subsets, then again E is the set of nonnegative bounded continuous mappings on Ω , but with $\tilde{\mathcal{S}}$ gathering all the functionally closed subsets. Then in view of Urysohn's lemma the assumptions (2.5), (2.6) are fulfilled for (Ω, τ_Ω) being normal. Thus the classical Portmanteau lemma may be extended to finite Borel-measures which are inner regular w.r.t. the closed subsets if (Ω, τ_Ω) is normal.

4. Compactness in spaces of inner regular measures

Let $\mathcal{S}, \tilde{\mathcal{S}}$ be lattices on a nonvoid set Ω with $\tilde{\mathcal{S}} \subseteq \mathcal{S}$, and satisfying the conditions (2.1)–(2.4). Furthermore let us retake the further notations from Section 2. We want to investigate necessary and sufficient conditions for compactness w.r.t. the weak topology τ_w on $\mathcal{M}_f(\Omega, \mathcal{S})$. Let us begin with the considerations under the assumptions (2.5)–(2.8).

Theorem 4.1. Let $cl(\Delta)$ be the closure of some nonvoid $\Delta \subseteq \mathcal{M}_f(\Omega, \mathcal{S})$ w.r.t. τ_w , let $\nu := \sup_{Q \in cl(\Delta)} Q$, and let $\mathcal{G}_{\tilde{\mathcal{S}}}$ consist of all sets $\Omega \setminus A$ with $A \in \tilde{\mathcal{S}}$. Additionally, let the assumptions (2.5)–(2.8) be satisfied, and consider the following statements:

- (1) Δ is relatively compact w.r.t. τ_w .
- (2) ν is real-valued, and $\nu|_{\tilde{\mathcal{S}}}$ is downward continuous.
- (3) ν is real-valued, and $\nu|_{\tilde{\mathcal{S}}}$ is downward continuous at \emptyset , satisfying $\nu(A) = \inf_{A \subseteq G \in \mathcal{G}_{\tilde{\mathcal{S}}}} \sup_{G \supseteq B \in \tilde{\mathcal{S}}} \nu(B)$ for each $A \in \tilde{\mathcal{S}}$.

Then the implications (3) \Rightarrow (1) \Rightarrow (2) are valid. Moreover, the statements (1)–(3) are equivalent if we may find for any $A \in \tilde{\mathcal{S}}$ a sequence $(A_n)_n$ in $\tilde{\mathcal{S}}$ with $\Omega \setminus A = \bigcup_{n=1}^\infty A_n$.

Remark. It is already known that the implication (3) \Rightarrow (2) is even valid when Δ is not relatively compact (cf. [15, Lemma 1.4]).

Proof. *Proof of (1) \Rightarrow (2):*

By definition the mapping $\psi_\Omega : cl(\Delta) \rightarrow \mathbb{R}, Q \mapsto Q(\Omega)$, is continuous w.r.t. the relative topology of τ_w to $cl(\Delta)$. Then, due to compactness of $cl(\Delta)$, the set $\{Q(\Omega) \mid Q \in cl(\Delta)\}$ is compact. In particular ν is real-valued.

Next, let $(A_n)_n$ be an antitone sequence in \mathcal{S} with $\bigcap_{n=1}^{\infty} A_n =: A_0 \in \mathcal{S}$. By definition of the weak topology the mappings

$$\psi_{A_n} : cl(\Delta) \rightarrow \mathbb{R}, \quad Q \mapsto Q(A_n) \quad (n \in \mathbb{N}_0)$$

are upper semicontinuous w.r.t. the relative topology of τ_w to $cl(\Delta)$. Since $cl(\Delta)$ is assumed to be a compact Hausdorff space w.r.t. the relative topology of τ_w , we may apply the general Dini lemma (cf. [9, Theorem 3.7]), and we obtain

$$\inf_{n \in \mathbb{N}} \nu(A_n) = \inf_{n \in \mathbb{N}} \sup_{Q \in cl(\Delta)} \psi_{A_n}(Q) = \sup_{Q \in cl(\Delta)} \inf_{n \in \mathbb{N}} \psi_{A_n}(Q) = \sup_{Q \in cl(\Delta)} \psi_{A_0}(Q) = \nu(A_0).$$

Proof of (3) \Rightarrow (1):

Since $(\mathcal{M}_f(\Omega, \mathcal{S}), \tau_w)$ is a regular Hausdorff space, it is known that Δ is relatively compact if and only if every universal net in Δ converges w.r.t. τ_w (cf. [21, Lemma 2.3]). So let us fix a universal net $(Q_j)_{j \in J}$ in Δ . It induces for each $A \in \tilde{\mathcal{S}}$ the universal net $(Q_j(A))_{j \in J}$ in \mathbb{R} and the relatively compact subset $\{Q_j(A) \mid j \in J\}$ since ν is real-valued. Therefore, we obtain some mapping $\phi : \tilde{\mathcal{S}} \rightarrow \mathbb{R}$ such that $\phi(A) = \lim_j Q_j(A)$ for every $A \in \tilde{\mathcal{S}}$. Routine procedures yield that ϕ is an isotone modular set function with $\phi(\emptyset) = 0$ and $\phi \leq \nu|_{\tilde{\mathcal{S}}}$.

By assumption on $\nu|_{\tilde{\mathcal{S}}}$, we have $\hat{\phi} \leq \nu|_{\tilde{\mathcal{S}}}$ for the isotone set function

$$\hat{\phi} : \tilde{\mathcal{S}} \rightarrow \mathbb{R}, \quad A \mapsto \inf_{A \subseteq G \in \mathcal{G}_{\tilde{\mathcal{S}}}} \sup_{G \supseteq B \in \tilde{\mathcal{S}}} \phi(B),$$

which even satisfies $\inf_n \hat{\phi}(A_n) = 0$ for any antitone sequence $(A_n)_n$ in $\tilde{\mathcal{S}}$ with $\bigcap_{n=1}^{\infty} A_n = \emptyset$ because $\nu|_{\tilde{\mathcal{S}}}$ satisfies this property. Thus, drawing on Proposition 1.1(2), we obtain $\hat{\phi}$ as an inner premeasure with $\phi(\Omega) = \hat{\phi}(\Omega)$. This means that there is some finite measure \hat{Q} on $\sigma(\tilde{\mathcal{S}})$ which is inner regular w.r.t. $\tilde{\mathcal{S}}$, and which satisfies $\hat{\phi} = \hat{Q}|_{\tilde{\mathcal{S}}}$. By Proposition 1.2, \hat{Q} may be extended to some $Q \in \mathcal{M}_f(\Omega, \mathcal{S})$. Then $\limsup_j Q_j(A) \leq Q(A)$ for $A \in \tilde{\mathcal{S}}$ due to $\phi \leq \hat{\phi}$, and $\lim_j Q_j(\Omega) = \phi(\Omega) = \hat{\phi}(\Omega)$. Therefore, in view of Theorem 2.3, the net $(Q_j)_{j \in J}$ converges to Q w.r.t. τ_w . Thus Δ is relatively compact w.r.t. τ_w .

Now let us assume that we may find for any $A \in \tilde{\mathcal{S}}$ a sequence $(A_n)_n$ in $\tilde{\mathcal{S}}$ with $\Omega \setminus A = \bigcup_{n=1}^{\infty} A_n$. It remains to prove the implication (2) \Rightarrow (3).

Proof of (2) \Rightarrow (3):

Let us fix $A \in \tilde{\mathcal{S}}$. By assumption there exists an isotone sequence $(A_n)_n$ in $\tilde{\mathcal{S}}$ with $\Omega \setminus A = \bigcup_{n=1}^{\infty} A_n$. Moreover, for each n we may find by (2.7) disjoint $G_{1n}, G_{2n} \in \mathcal{G}_{\tilde{\mathcal{S}}}$ with $A \subseteq G_{1n}$ and $A_n \subseteq G_{2n}$. Then we can define by $G_n := \bigcap_{m=1}^n G_{1m}$ and $B_n := \bigcap_{m=1}^n \Omega \setminus G_{2m}$ antitone sequences $(G_n)_n$ and $(B_n)_n$ in $\mathcal{G}_{\tilde{\mathcal{S}}}$ and $\tilde{\mathcal{S}}$ respectively with $A \subseteq G_n \subseteq B_n \subseteq \Omega \setminus A_n$, implying $A = \bigcap_{n=1}^{\infty} G_n = \bigcap_{n=1}^{\infty} B_n$. Hence we may conclude from statement (2)

$$\nu(A) = \lim_{n \rightarrow \infty} \nu(B_n) \geq \lim_{n \rightarrow \infty} \sup_{G_n \supseteq B \in \tilde{\mathcal{S}}} \nu(B) \geq \inf_{A \subseteq G \in \mathcal{G}_{\tilde{\mathcal{S}}}} \sup_{G \supseteq B \in \tilde{\mathcal{S}}} \nu(B) \geq \nu(A).$$

Since $\nu|_{\tilde{\mathcal{S}}}$ is downward continuous at \emptyset by statement (2) again, statement (3) is shown, which completes the proof. \square

Drawing on Topsoe's investigations in [19] we may give a further characterization of relatively compact subsets in the topological subspace $\mathcal{M}_f(\Omega, \mathcal{S}, \tau)$ consisting of all $Q \in \mathcal{M}_f(\Omega, \mathcal{S})$ with $\inf_{A \in M} Q(A) = Q(B)$ for every nonvoid downward directed family $M \subseteq \mathcal{S}$ with $\bigcap_{A \in M} A = B \in \mathcal{S}$.

Theorem 4.2. *Let $cl(\Delta)$ be the closure of some nonvoid $\Delta \subseteq \mathcal{M}_f(\Omega, \mathcal{S}, \tau)$, let $\nu := \sup_{Q \in cl(\Delta)} Q$, and let $\bar{\nu} := \sup_{Q \in \Delta} Q$. If for any $A \in \mathcal{S}$ and every $\omega \in \Omega \setminus A$ there is some $B \in \tilde{\mathcal{S}}$ with $\omega \in B \subseteq \Omega \setminus A$, then under the assumptions (2.5)–(2.8) the following statements are equivalent:*

- (1) Δ is relatively compact w.r.t. the relative topology of τ_w to $\mathcal{M}_f(\Omega, \mathcal{S}, \tau)$.
- (2) ν is real-valued, and $\inf_{A \in M} \nu(A) = \nu(B)$ holds for any nonvoid downward directed family $M \subseteq \mathcal{S}$ with $\bigcap_{A \in M} A = B \in \mathcal{S}$.
- (3) $\nu(\Omega) < \infty$, and $\inf_{A \in M} \nu(A) = 0$ for each nonvoid downward directed family $M \subseteq \mathcal{S}$ with $\bigcap_{A \in M} A = \emptyset$.
- (4) $\bar{\nu}(\Omega) < \infty$, and $\inf_{A \in M} \bar{\nu}(A) = 0$ for each nonvoid downward directed family $M \subseteq \mathcal{S}$ with $\bigcap_{A \in M} A = \emptyset$.

Proof. Let the assumptions (2.5)–(2.8) be valid. The implications (2) \Rightarrow (3) \Rightarrow (4) are trivial. Furthermore $\mathcal{M}_f(\Omega, \mathcal{S}, \tau)$, endowed with the relative topology of τ_w , is a regular Hausdorff space due to Proposition 2.4. Therefore, due to Lemma 2.3 in [21], Δ is relatively compact w.r.t. the relative topology of τ_w to $\mathcal{M}_f(\Omega, \mathcal{S}, \tau)$ if and only if every universal net in Δ converges in $\mathcal{M}_f(\Omega, \mathcal{S}, \tau)$ w.r.t. this topology.

Proof of (1) \Rightarrow (2):

In view of statement (1) the topological closure of Δ in $\mathcal{M}_f(\Omega, \mathcal{S}, \tau)$ w.r.t. to the relative topology of τ_w is a compact subset w.r.t. τ_w , which means that it is also closed, thus enclosing $cl(\Delta)$. In particular $cl(\Delta)$ is a compact subset w.r.t. τ_w and a subset of $\mathcal{M}_f(\Omega, \mathcal{S}, \tau)$.

By definition the mapping $\psi_\Omega : cl(\Delta) \rightarrow \mathbb{R}$, $Q \mapsto Q(\Omega)$, is continuous w.r.t. the relative topology of τ_w to $cl(\Delta)$. Then, due to compactness of $cl(\Delta)$, the set $\{Q(\Omega) \mid Q \in cl(\Delta)\}$ is compact. In particular ν is real-valued.

Next, let $M \subseteq \mathcal{S}$ denote a nonvoid downward directed family with $\bigcap_{B \in M} B =: A \in \mathcal{S}$. By definition of the weak topology the mappings

$$\psi_B : cl(\Delta) \rightarrow \mathbb{R}, \quad Q \mapsto Q(B) \quad (B \in M \cup \{A\})$$

are upper semicontinuous w.r.t. the relative topology of τ_w to $cl(\Delta)$. Since $cl(\Delta)$ is a compact Hausdorff space w.r.t. the relative topology of τ_w , we may apply the general Dini lemma (cf. [9, Theorem 3.7]) to obtain

$$\inf_{B \in M} \nu(B) = \inf_{B \in M} \sup_{Q \in cl(\Delta)} \psi_B(Q) = \sup_{Q \in cl(\Delta)} \inf_{B \in M} \psi_B(Q) = \sup_{Q \in cl(\Delta)} \psi_A(Q) = \nu(A).$$

Proof of (4) \Rightarrow (1):

We have to show that every universal net in Δ converges in $\mathcal{M}_f(\Omega, \mathcal{S}, \tau)$ w.r.t. the relative topology of τ_w . So let $(P_j)_{j \in J}$ denote some universal net in Δ . Analogously to the proof of the implication (3) \Rightarrow (1) of Theorem 4.1, we obtain an isotone modular set function $\phi : \mathcal{G}_{\tilde{\mathcal{S}}} \rightarrow [0, \infty[$ with $\phi(G) = \lim_j P_j(G)$ for any $G \in \mathcal{G}_{\tilde{\mathcal{S}}}$. In particular $\phi \leq \bar{\nu}|_{\mathcal{G}_{\tilde{\mathcal{S}}}}$. Then define now

$$\hat{\phi} : \tilde{\mathcal{S}} \rightarrow [0, \infty[, \quad A \mapsto \inf_{A \subseteq G \in \mathcal{G}_{\tilde{\mathcal{S}}}} \phi(G).$$

Due to (2.7) we may apply Lemmata 1, 2 in [19] to verify $\hat{\phi}$ as an isotone modular set function satisfying

$$\hat{\phi}(A_2) = \sup_{A_2 \setminus A_1 \supseteq B \in \tilde{\mathcal{S}}} \hat{\phi}(B) + \hat{\phi}(A_1)$$

for $A_1, A_2 \in \tilde{\mathcal{S}}$ with $A_1 \subseteq A_2$. In view of Proposition 3.3 in [13] this implies

$$\hat{\phi}(A) = \inf_{A \subseteq G \in \mathcal{G}_{\tilde{\mathcal{S}}}} \sup_{G \supseteq B \in \tilde{\mathcal{S}}} \hat{\phi}(B) \quad \text{for any } A \in \tilde{\mathcal{S}}.$$

Furthermore let $M \subseteq \tilde{\mathcal{S}}$ be nonvoid and downward directed with $\bigcap_{B \in M} B = \emptyset$, and define \tilde{M} to consist of all $A \in \tilde{\mathcal{S}}$ with $B \subseteq G \subseteq A$ for some $B \in M$ and $G \in \mathcal{G}_{\tilde{\mathcal{S}}}$. This family is nonvoid, downward directed, and by the added assumption of separation for τ_w as well as condition (2.7) $\bigcap_{A \in \tilde{M}} A = \emptyset$ may be concluded. Then for any $\varepsilon > 0$, the application of statement (4) gives some $A \in \tilde{M}$ with $\bar{\nu}(A) < \varepsilon$. By construction, there is some $B \in M$ and some $G \in \mathcal{G}_{\tilde{\mathcal{S}}}$ with $B \subseteq G \subseteq A$. Hence

$$\inf_{B \in M} \hat{\phi}(B) \leq \phi(G) = \lim_j P_j(G) \leq \bar{\nu}(G) \leq \bar{\nu}(A) < \varepsilon.$$

Therefore

$$(*) \inf_{B \in M} \hat{\phi}(B) = 0 \text{ for any nonvoid downward directed family } M \subseteq \tilde{\mathcal{S}} \text{ with } \bigcap_{B \in M} B = \emptyset.$$

In particular $\hat{\phi}$ is a premeasure in view of Proposition 1.1(2). Drawing on Proposition 1.2, we may find some $P \in \mathcal{M}_f(\Omega, \mathcal{S})$ with $P|_{\tilde{\mathcal{S}}} = \hat{\phi}$. Moreover, for any $A \in \tilde{\mathcal{S}}$ and $\varepsilon > 0$, there exists some $G \in \mathcal{G}_{\tilde{\mathcal{S}}}$ with $A \subseteq G$ and $P(A) = \hat{\phi}(A) > \phi(G) - \varepsilon$. This implies by construction

$$P(A) + \varepsilon \geq \limsup_j P_j(G) \geq \limsup_j P_j(A)$$

and

$$P(\Omega) = \hat{\phi}(\Omega) = \phi(\Omega) = \lim_j P_j(\Omega).$$

Hence $(P_j)_{j \in J}$ converges to P w.r.t. τ_w by Theorem 2.3. It remains to show $P \in \mathcal{M}_f(\Omega, \mathcal{S}, \tau)$. Since

$$P(A_2) = \sup_{A_2 \setminus A_1 \supseteq B \in \mathcal{S}} P(B) + P(A_1)$$

for $A_1, A_2 \in \mathcal{S}$ with $A_1 \subseteq A_2$, it suffices to prove $\inf_{B \in M} P(B) = 0$ for any nonvoid downward directed family $M \subseteq \mathcal{S}$ with $\bigcap_{B \in M} B = \emptyset$ (cf. [19, Lemma 1]). Let $M \subseteq \mathcal{S}$ denote such a family, and define N to consist of all $B \in \tilde{\mathcal{S}}$ with $D \subseteq B$ for some $D \in M$, which contains at least Ω . Furthermore, N is downward directed since M satisfies this property. Moreover, for any $\omega \in \Omega$ there is some $D \in M$ with $\omega \in \Omega \setminus D$. By assumption we may find a set $B \in \tilde{\mathcal{S}}$ with $\omega \in B \subseteq \Omega \setminus D$, which implies by condition (2.6) that there is a set $\hat{B} \in \tilde{\mathcal{S}}$ with $D \subseteq \hat{B} \subseteq \Omega \setminus B$. Thus $\hat{B} \in N$ as well as $\omega \in B \subseteq \Omega \setminus \hat{B}$, and therefore $\Omega = \bigcup_{B \in N} \Omega \setminus B = \Omega \setminus \bigcap_{B \in N} B$.

Finally, $\inf_{A \in N} P(A) = 0$ due to $(*)$, and thus $\inf_{B \in M} P(B) = 0$ by construction of N , which completes the proof. \square

In the following we want to use the general Portmanteau Lemma 3.1 to obtain a result concerning compactness w.r.t. the weak topology.

Theorem 4.3. Let $E \subseteq \{X \in [0, \infty]^{\Omega} \mid X \text{ bounded}\}$ denote a Stonean lattice cone with $1 \in E$, let further $L := E - E$ and $\tilde{S} := \{\bigcap_{n=1}^{\infty} X_n^{-1}([x_n, \infty]) \mid X_n \in E, x_n > 0\}$. Furthermore $\text{cl}(\Delta)$ denotes the closure of some subset Δ of $\mathcal{M}_f(\Omega, \mathcal{S})$ w.r.t. τ_w , and induces the mappings $\nu := \sup_{Q \in \text{cl}(\Delta)} Q$ as well as $I : L \rightarrow]-\infty, \infty]$, which is defined by $I(X) = \sup_{Q \in \text{cl}(\Delta)} \int X dQ$.

Then under the assumptions (2.5), (2.6), (2.8), (3.1) $I(X) = \sup_{Q \in \Delta} \int X dQ$ for $X \in L$, and the following statements are equivalent:

- (1) Δ is relatively compact w.r.t. τ_w .
- (2) I is real-valued, and $I(X_n) \searrow I(X)$ whenever $(X_n)_n$ is an antitone sequence in L with $X_n \searrow X \in L$.

Furthermore each of the statements (1), (2) implies

- (3) ν is real-valued, and $\nu|_{\tilde{S}}$ is downward continuous.

If in addition assumption (2.7) is satisfied, then the statements (1)–(3) are equivalent, and each of them is equivalent with

- (4) ν is real-valued, and $\nu|_{\tilde{S}}$ is downward continuous at \emptyset with $\nu(A) = \inf_{A \subseteq G \in \mathcal{G}_{\tilde{S}}} \sup_{G \supseteq B \in \tilde{S}} \nu(B)$ for each $A \in \tilde{S}$.

Proof. Firstly, Portmanteau Lemma 3.1 means that $\tau_w = \tau_{w,E}$ under (2.5), (2.6) and (3.1). Hence in this situation $I(X) = \sup_{Q \in \Delta} \int X dQ$ holds for each $X \in L$.

Furthermore, due to (3.1), for any $A_0 \in \tilde{S}$ we have $\Omega \setminus A_0 = \bigcup_{n=1}^{\infty} A_n$ for some sequence $(A_n)_{n \in \mathbb{N}}$ in \tilde{S} . Thus the equivalence of (1), (3), (4) in the case of (2.5)–(2.8) follows immediately from Theorem 4.1, whereas under (2.5), (2.6), (3.1) τ_w is a completely regular Hausdorff space due to Proposition 3.2, and the implication (1) \Rightarrow (3) can be shown as in the proof of Theorem 4.1. Therefore it remains to prove equivalence of (1) and (2) if (2.5), (2.6), (2.8) and (3.1) are valid.

Proof of (1) \Rightarrow (2):

For every $X \in L$ the mapping $\psi_X : \mathcal{M}_f(\Omega, \mathcal{S}) \rightarrow \mathbb{R}, Q \mapsto \int X dQ$ is continuous w.r.t. the weak topology due to the general Portmanteau lemma. Therefore I is real-valued by statement (1). Let $(X_n)_n$ be an antitone sequence in L with $X_n \searrow X$ for some $X \in L$. Then the general Dini lemma (cf. [9, Theorem 3.7]) yields

$$I(X) = \sup_{Q \in \text{cl}(\Delta)} \psi_X(Q) = \sup_{Q \in \text{cl}(\Delta)} \inf_n \psi_{X_n}(Q) = \inf_n \sup_{Q \in \text{cl}(\Delta)} \psi_{X_n}(Q) = \inf_n I(X_n),$$

which shows statement (2).

Proof of (2) \Rightarrow (1):

Let L^* denote the space of real linear forms on L . It will be equipped with the so called weak * topology, i.e. the relative topology of the product topology on \mathbb{R}^L to L^* .

By assumption I is a real sublinear form on L , and it is therefore associated with the nonvoid set $\Delta(I)$ consisting of all $\Lambda \in L^*$ with $\Lambda \leq I$. It is known that $\Delta(I)$ is a compact subset w.r.t. the weak * topology. This follows from $\Delta(I) \subseteq L^* \cap \bigcap_{X \in L} [-I(-X), I(X)]$. This description also ensures that every linear form from $\Delta(I)$ is positive, in particular the restrictions to E are isotone and positive-linear.

Let $(Q_j)_{j \in J}$ be a net in Δ . Then, defining $\Lambda_{Q_j} \in L^*$ by $\Lambda_{Q_j}(X) = \int X dQ_j$, we obtain by $(\Lambda_{Q_j})_{j \in J}$ a net in $\Delta(I)$. Compactness of $\Delta(I)$ implies that there is a subnet $(\Lambda_{Q_{j(k)}})_{k \in K}$ which converges to some $\Lambda \in \Delta(I)$ w.r.t. the weak * topology. Moreover, statement (2) ensures that $\lim_n \Lambda(X_n) = 0 = \lim_n \Lambda(Y - Y_n)$ holds for any antitone sequence $(X_n)_n$ in E with $X_n \searrow 0$ and every isotone sequence $(Y_n)_n$ in E with $Y_n \nearrow Y \in E$. Thus we may apply the inner Daniell–Stone Theorem 1.3 to $\Lambda|_E$. Hence we can find some finite measure P on $\sigma(\tilde{S})$ which is inner regular w.r.t. \tilde{S} and satisfies $\Lambda(X) = \int X dP$ for $X \in E$. Drawing on Proposition 1.2, P can be extended uniquely to some finite measure $Q \in \mathcal{M}_f(\Omega, \mathcal{S})$, so that $\Lambda(X) = \int X dQ$ holds for $X \in E$. In particular $\lim_k \int X dQ_{j(k)} = \int X dQ$ for every $X \in E$. That means that $(Q_{j(k)})_{k \in K}$ converges to Q w.r.t. the weak topology due to the general Portmanteau lemma. Thus Δ is relatively compact, which completes the proof. \square

In the following we shall present a criterion to replace condition (2.7), which implies the following variant of Theorem 4.3.

Corollary 4.4. Let us retake notations and assumptions from Theorem 4.3. If $\frac{X}{Y} \in E$ holds for $X, Y \in E$ with $Y > 0$, and if $\sum_{n=1}^{\infty} \frac{X_n}{2^n} \in E$ whenever $(X_n)_n$ is a uniformly bounded sequence in E , then under the assumptions (2.5), (2.6), (2.8) and (3.1) all the four statements from Theorem 4.3 are equivalent.

Proof. It remains to prove that assumption (2.7) is valid. For this purpose let $A_1, A_2 \in \tilde{S}$ be disjoint. By definition there exist antitone sequences $(X_{1n})_n, (X_{2n})_n$ in E as well as sequences $(x_{1n})_n, (x_{2n})_n$ of positive numbers with $A_i = \bigcap_{n=1}^{\infty} X_{ni}^{-1}([x_{ni}, \infty])$ for $i = 1, 2$. Then $Z_{ni} := 1 - \frac{\min\{X_{ni}, x_{ni}\}}{x_{ni}}$ belongs to E with $0 \leq Z_{ni} \leq 1$ for $i = 1, 2, n \in \mathbb{N}$. By assumption $Z_i := \sum_{n=1}^{\infty} \frac{Z_{ni}}{2^n} \in E$, satisfying $A_i = Z_i^{-1}(\{0\})$ for $i = 1, 2$. Since A_1, A_2 are disjoint, $Z_1 + Z_2 \in E$ with $Z_1 + Z_2 > 0$. Hence by assumption

$Y_i := \frac{Z_i}{Z_1 + Z_2}$ is a member of E for $i = 1, 2$. Thus, $Y_i^{-1}([0, \frac{1}{4}])$ ($i = 1, 2$) are disjoint elements of $\mathcal{G}_{\tilde{\mathcal{S}}}$ with $A_i \subseteq Y_i^{-1}([0, \frac{1}{4}])$ for $i = 1, 2$, which completes the proof. \square

Corollary 4.4 is useful to characterize relatively compact subsets of finite Baire-measures, and finite Borel-measures which are inner regular w.r.t. closed subsets. Let us start with relatively compact subsets of finite Baire-measures on Hausdorff spaces. We may apply Corollary 4.4 directly.

Corollary 4.5. Let τ_Ω be a Hausdorff topology on Ω , and let \mathcal{S}, \mathcal{T} be respectively the sets of functionally closed and functionally open subsets w.r.t. τ_Ω . Furthermore $cl(\Delta)$ denotes the closure of a set Δ of finite Baire-measures w.r.t. the topology generated by weak convergence, and induces the mapping $\nu := \sup_{Q \in cl(\Delta)} Q$. Additionally let L consist of all bounded real-valued continuous mappings on Ω , and let $I : L \rightarrow]-\infty, \infty]$ be defined by $I(X) = \sup_{Q \in cl(\Delta)} \int X dQ$.

Then the following statements are equivalent:

- (1) Δ is relatively compact w.r.t. the topology induced by weak convergence.
- (2) ν is real-valued, and $\nu|_{\mathcal{S}}$ is downward continuous.
- (3) ν is real-valued, and $\nu|_{\mathcal{S}}$ is downward continuous at \emptyset with $\nu(A) = \inf_{A \subseteq G \in \mathcal{T}} \sup_{G \supseteq B \in \mathcal{S}} \nu(B)$ for each $A \in \mathcal{S}$.
- (4) I is real-valued, and $\lim_{n \rightarrow \infty} I(X_n) = I(X)$ if $(X_n)_n$ is an antitone sequence in L with $X_n \searrow X \in L$.

Remark. Varadarajan has already shown the equivalence of the statements (1), (4) in Corollary 4.5 (cf. [22, Theorem 25]).

In order to apply Corollary 4.4 to finite Borel-measures let us consider a normal and countably paracompact topology on Ω . For instance perfectly normal and metrizable topologies satisfy these properties (cf. [5, 5.2.5, 4.1.13]). Then we may choose for \mathcal{S} the lattice of all closed subsets, and for $\tilde{\mathcal{S}}$ the set of all functionally closed subsets. Additionally E is defined to consist of all nonnegative bounded real-valued continuous mappings on Ω . Noticing Urysohn's lemma, $\mathcal{S}, \tilde{\mathcal{S}}$ and E satisfy the requirements of Corollary 4.4 to guarantee the equivalence of all the statements there. Then Theorem 4.3 reads as follows.

Corollary 4.6. Let (Ω, τ_Ω) be a normal and countably paracompact space, and let $\mathcal{S}, \tilde{\mathcal{S}}, \mathcal{T}$ be respectively the set of closed, functionally closed and functionally open subsets w.r.t. τ_Ω . Furthermore Δ denotes a set of finite Borel-measures which are inner regular w.r.t. the closed subsets, and let $cl(\Delta)$ be the closure of Δ w.r.t. the topology generated by weak convergence. Additionally, let L consist of all bounded real-valued continuous mappings on Ω , and let $I : L \rightarrow]-\infty, \infty]$ be defined by $I(X) = \sup_{Q \in cl(\Delta)} \int X dQ$.

Then, setting $\nu := \sup_{Q \in cl(\Delta)} Q$, the following statements are equivalent:

- (1) Δ is relatively compact w.r.t. the topology induced by weak convergence.
- (2) ν is real-valued, and $\nu|_{\tilde{\mathcal{S}}}$ is downward continuous.
- (3) ν is real-valued, and $\nu|_{\tilde{\mathcal{S}}}$ is downward continuous at \emptyset with $\nu(A) = \inf_{A \subseteq G \in \mathcal{T}} \sup_{G \supseteq B \in \tilde{\mathcal{S}}} \nu(B)$ for each $A \in \tilde{\mathcal{S}}$.
- (4) I is real-valued, and $\lim_{n \rightarrow \infty} I(X_n) = I(X)$ if $(X_n)_n$ is an antitone sequence in L with $X_n \searrow X \in L$.

Remark. Corollary 4.6 encompasses the case that (Ω, τ_Ω) is perfectly normal. Since in perfectly normal spaces all closed subsets are functionally closed, and each open subset is functionally open (cf. [5, 1.5.19]), we may replace then in Corollary 4.6 $\tilde{\mathcal{S}}$ by \mathcal{S} and \mathcal{T} by τ_Ω . Moreover, Corollary 4.6 generalizes also a result by Huber and Strassen who showed the equivalence of statements (1), (2) in Corollary 4.6 for probability measures on Polish spaces [8]. Note that metrizable topologies are perfectly normal.

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